# More trigonometry – MS

**1.** *[6 marks]*

Show that $\frac{\cos A+\sin A}{\cos A-\sin A} = \sec 2A + \tan 2A$.

## Markscheme

**METHOD 1**

$\frac{\cos A+\sin A}{\cos A-\sin A} = \sec 2A + \tan 2A$

consider right hand side

$\sec 2A + \tan 2A = \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A}$ ***M1A1***

$= \frac{\cos^2 A+2\sin A\cos A+\sin^2 A}{\cos^2 A-\sin^2 A}$ ***A1A1***

**Note:** Award ***A1*** for recognizing the need for single angles and ***A1*** for recognizing $\cos^2 A + \sin^2 A = 1$.

$= \frac{(\cos A+\sin A)^2}{(\cos A+\sin A)(\cos A-\sin A)}$ ***M1A1***

$= \frac{\cos A+\sin A}{\cos A-\sin A}$ ***AG***

**METHOD 2**

$\frac{\cos A+\sin A}{\cos A-\sin A} = \frac{(\cos A+\sin A)^2}{(\cos A+\sin A)(\cos A-\sin A)}$ ***M1A1***

$= \frac{\cos^2 A+2\sin A\cos A+\sin^2 A}{\cos^2 A-\sin^2 A}$ ***A1A1***

**Note:** Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{1+\sin 2A}{\cos 2A}$ ***M1A1***

$= \sec 2A + \tan 2A$ ***AG***

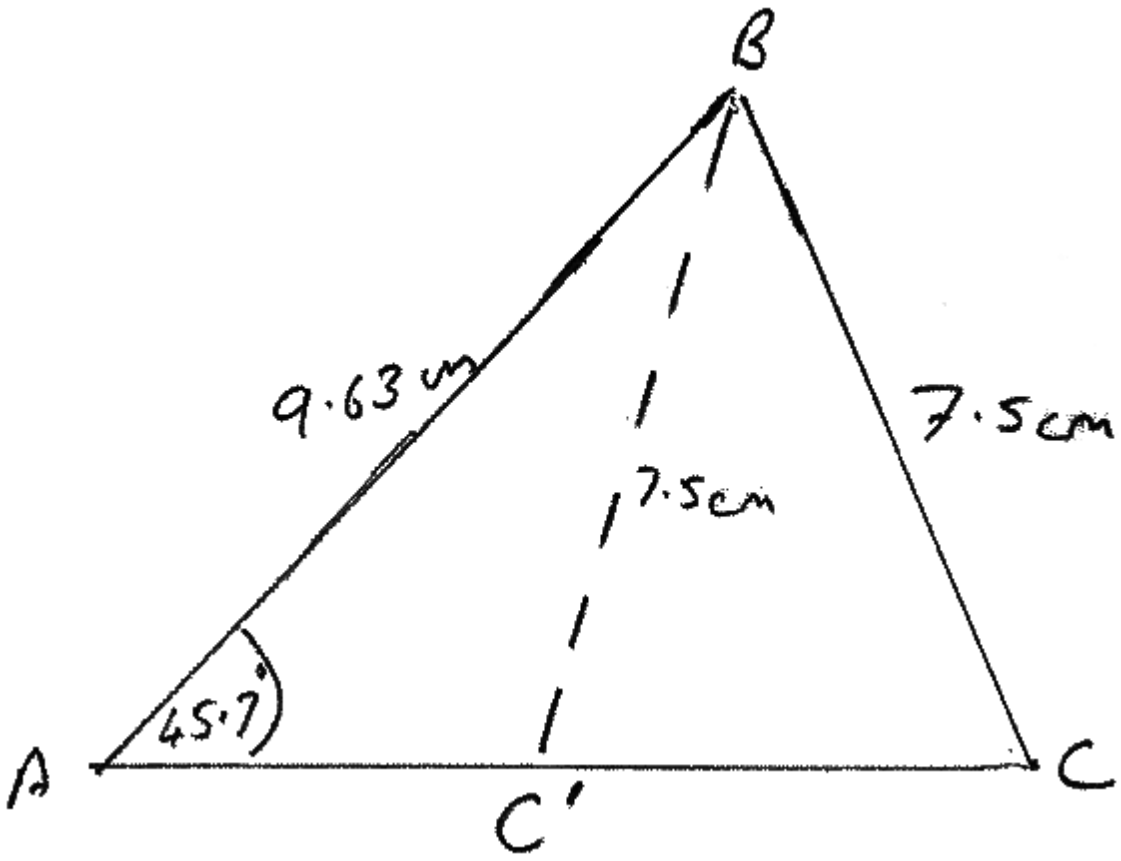
***[6 marks]***

**2a.** *[2 marks]*

Consider a triangle ABC with $\mathrm{B\hat{A}C} = 45.7^\circ$ , AB = 9.63 cm and BC = 7.5 cm .

By drawing a diagram, show why there are two triangles consistent with this information.

Markscheme




***A2***

**Note:** Accept 2 separate triangles. The diagram(s) should show that one triangle has an acute angle and the other triangle has an obtuse angle. The values 9.63, 7.5 and 45.7 and/or the letters, A, B, C′ and C should be correctly marked on the diagram(s).

***[2 marks]***

**2b.** *[6 marks]*

Find the possible values of AC .

Markscheme

**METHOD 1**

$\frac{\sin 45.7}{7.5} = \frac{\sin C}{9.63}$ ***M1***

$\Rightarrow \hat{C} = 66.77...^\circ\ ,\ 113.2...^\circ$ ***(A1)(A1)***

$\Rightarrow \hat{B} = 67.52...^\circ\ ,\ 21.07...^\circ$ ***(A1)***

$\frac{b}{\sin B} = \frac{7.5}{\sin 45.7} \Rightarrow b = 9.68(\text{cm}), b = 3.77(\text{cm})$ ***A1A1***

**Note:** If only the acute value of $\hat{C}$ is found, award ***M1(A1)(A0)(A0)A1A0***.

**METHOD 2**

$7.5^2 = 9.63^2 + b^2 - 2 \times 9.63 \times b\cos 45.7^\circ$ ***M1A1***

$b^2 - 13.45...b + 36.48... = 0$

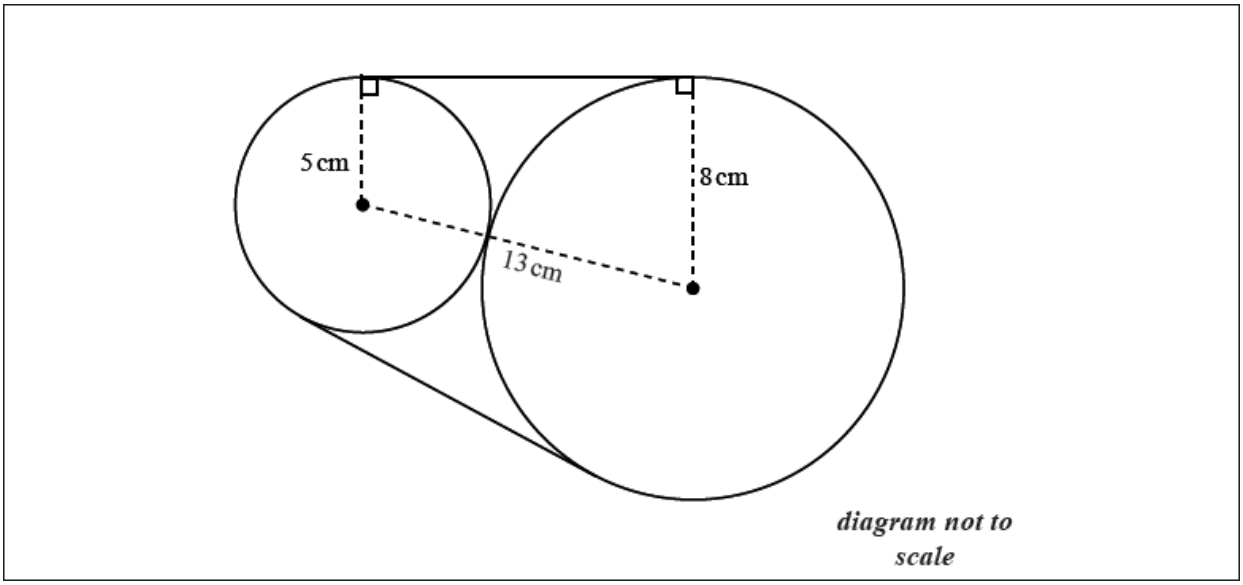
$b = \frac{13.45... \pm \sqrt{13.45...^2 - 4 \times 36.48...}}{2}$ ***(M1)(A1)***

$\text{AC} = 9.68(\text{cm})\ ,\ \text{AC} = 3.77(\text{cm})$ ***A1A1***

***[6 marks]***

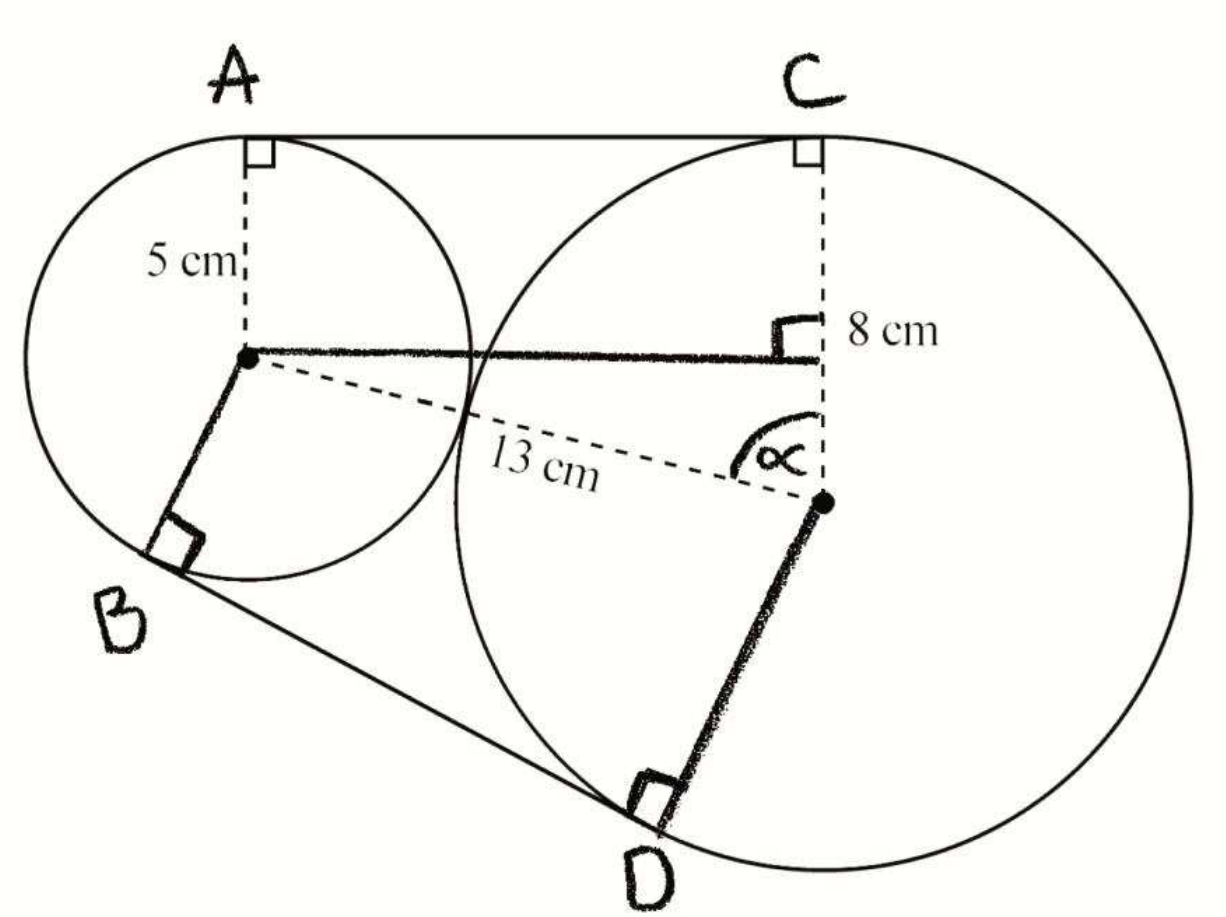
**3.** *[8 marks]*

Two discs, one of radius 8 cm and one of radius 5 cm, are placed such that they touch each other. A piece of string is wrapped around the discs. This is shown in the diagram below.




Calculate the length of string needed to go around the discs.

Markscheme




$\text{AC} = \text{BD} = \sqrt{13^2 - 3^2} = 12.64...$ ***(A1)***

$\cos\alpha = \frac{3}{13} \Rightarrow \alpha = 1.337...(76.65...^\circ.)$ ***(M1)(A1)***

attempt to find either arc length AB or arc length CD ***(M1)***

$\text{arc length AB} = 5(\pi - 2 \times 0.232...)\ (= 13.37...)$ ***(A1)***

$\text{arc length CD} = 8(\pi + 2 \times 0.232...)\ (= 28.85...)$ ***(A1)***

length of string = 13.37... + 28.85... + 2(12.64...) ***(M1)***

= 67.5 (cm) ***A1***

***[8 marks]***

**4.** *[4 marks]*

Given that $\frac{\pi}{2} < \alpha < \pi$ and $\cos\alpha = -\frac{3}{4}$, find the value of sin 2α .

Markscheme

$\sin\alpha = \sqrt{1-\left(-\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$ ***(M1)A1***

attempt to use double angle formula ***M1***

$\sin 2\alpha = 2\,\frac{\sqrt{7}}{4}\left(-\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$ ***A1***

**Note:** $\frac{\sqrt{7}}{4}$ seen would normally be awarded ***M1A1***.

***[4 marks]***

**5a.** *[4 marks]*

In the triangle PQR, PQ = 6 , PR = *k* and $\mathrm{P\hat{Q}R} = 30^\circ$ .

For the case *k* = 4 , find the two possible values of QR.

Markscheme

attempt to apply cosine rule ***M1***

$4^2 = 6^2 + \mathrm{QR}^2 - 2\cdot\mathrm{QR}\cdot 6\cos 30^\circ$ ( **or** $\mathrm{QR}^2 - 6\sqrt{3}\,\mathrm{QR} + 20 = 0$ ) ***A1***

$\mathrm{QR} = 3\sqrt{3} + \sqrt{7}$ **or** $\mathrm{QR} = 3\sqrt{3} - \sqrt{7}$ ***A1A1***

***[4 marks]***

**5b.** *[3 marks]*

Determine the values of *k* for which the conditions above define a unique triangle.

Markscheme

**METHOD 1**

$k \geqslant 6$ ***A1***

$k = 6\sin 30^\circ = 3$ ***M1A1***

**Note:** The ***M1*** in (b) is for recognizing the right-angled triangle case.

**METHOD 2**

$k \geqslant 6$ ***A1***

use of discriminant: $108 - 4(36 - k^2) = 0$ ***M1***

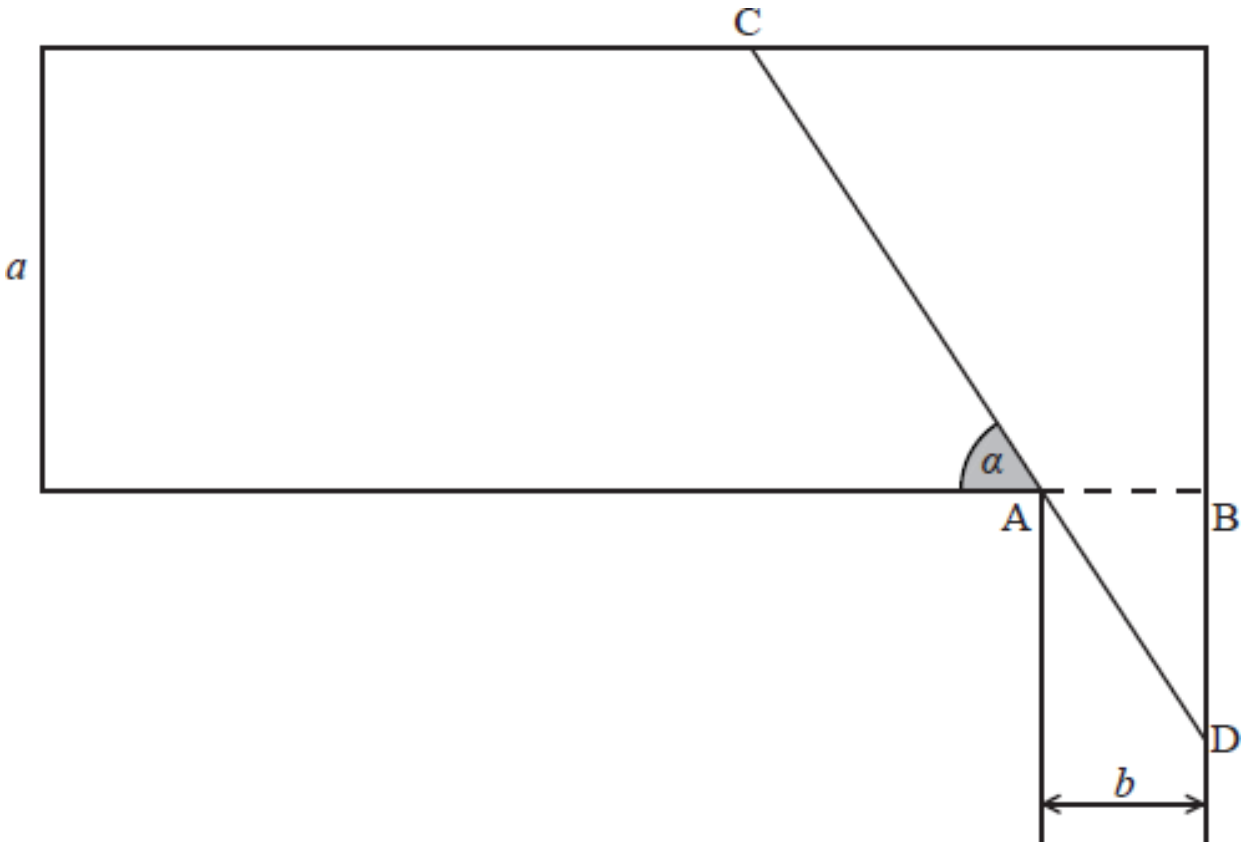
$k = 3$ ***A1***

**Note:** $k = \pm 3$ is ***M1A0***.

***[3 marks]***

**6.** *[3 marks]*

The diagram shows the plan of an art gallery *a* metres wide. [AB] represents a doorway, leading to an exit corridor *b* metres wide. In order to remove a painting from the art gallery, CD (denoted by *L* ) is measured for various values of $\alpha$ , as represented in the diagram.




If $\alpha$ is the angle between [CD] and the wall, show that $L = \frac{a}{\sin \alpha} + \frac{b}{\cos \alpha}$ , $0 < \alpha < \frac{\pi}{2}$.

Markscheme

$L = \text{CA} + \text{AD}$ ***M1***

$\sin\alpha = \frac{a}{\text{CA}} \Rightarrow \text{CA} = \frac{a}{\sin\alpha}$ ***A1***

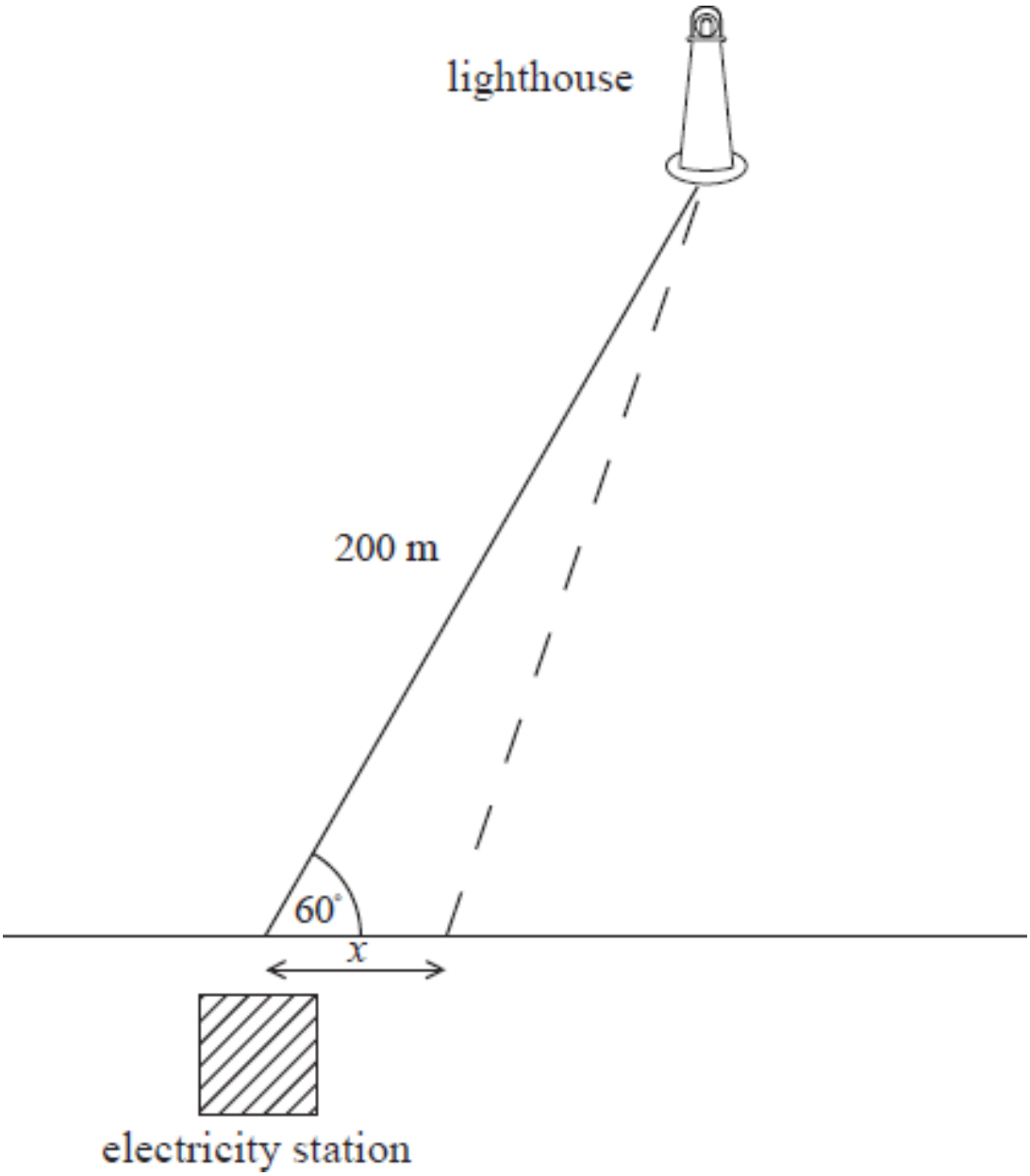
$\cos\alpha = \frac{b}{\text{AD}} \Rightarrow \text{AD} = \frac{b}{\cos\alpha}$ ***A1***

$L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$ ***AG***

***[2 marks]***

**7.** *[4 marks]*

An electricity station is on the edge of a straight coastline. A lighthouse is located in the sea 200 m from the electricity station. The angle between the coastline and the line joining the lighthouse with the electricity station is 60°. A cable needs to be laid connecting the lighthouse to the electricity station. It is decided to lay the cable in a straight line to the coast and then along the coast to the electricity station. The length of cable laid along the coastline is *x* metres. This information is illustrated in the diagram below.




The cost of laying the cable along the sea bed is US $80 per metre, and the cost of laying it on land is US $20 per metre.

Find, in terms of *x*, an expression for the cost of laying the cable.

## Markscheme

let the distance the cable is laid along the seabed be *y*

$y^2 = x^2 + 200^2 - 2 \times x \times 200 \cos 60^\circ$ ***(M1)***

(or equivalent method)

$y^2 = x^2 - 200x + 40000$ ***(A1)***

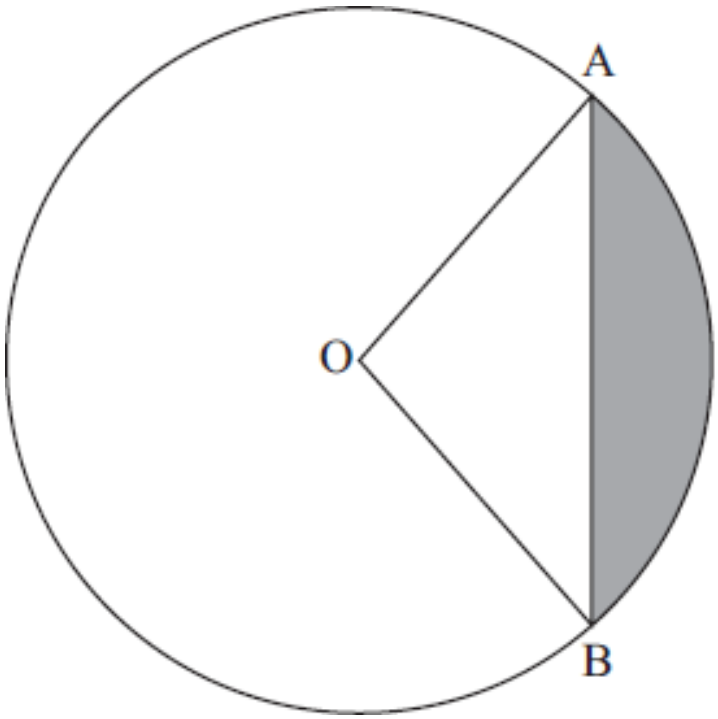
cost = $C = 80y + 20x$ ***(M1)***

$C = 80(x^2 - 200x + 40000)^{\frac{1}{2}} + 20x$ ***A1***

***[4 marks]***

**8a.** *[2 marks]*

A circle of radius 4 cm , centre O , is cut by a chord [AB] of length 6 cm.




Find $\mathrm{A\hat{O}B}$, expressing your answer in radians correct to four significant figures.

Markscheme

**EITHER**

$\mathrm{A\hat{O}B} = 2\arcsin\left(\frac{3}{4}\right)$ or equivalent (*eg* $\mathrm{A\hat{O}B} = 2\arctan\left(\frac{3}{\sqrt{7}}\right)$, $\mathrm{A\hat{O}B} = 2\arccos\left(\frac{\sqrt{7}}{4}\right)$) ***(M1)***

**OR**

$\cos \mathrm{A\hat{O}B} = \frac{4^2+4^2-6^2}{2\times4\times4}$ $\left(= -\frac{1}{8}\right)$ ***(M1)***

**THEN**

$= 1.696$ (correct to 4sf) ***A1***

***[2 marks]***

**8b.** *[3 marks]*

Determine the area of the shaded region.

## Markscheme

use of area of segment = area of sector – area of triangle ***(M1)***

$= \frac{1}{2} \times 4^2 \times 1.696 - \frac{1}{2} \times 4^2 \times \sin 1.696$ ***(A1)***

$= 5.63\ (\text{cm}^2)$ ***A1***

***[3 marks]***

**9.** *[6 marks]*

From a vertex of an equilateral triangle of side $2x$, a circular arc is drawn to divide the triangle into two regions, as shown in the diagram below.

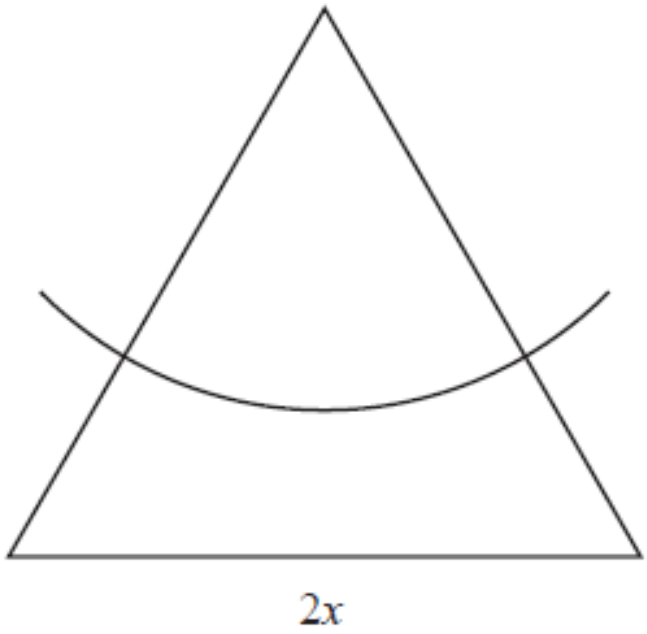



***diagram not to scale***

Given that the areas of the two regions are equal, find the radius of the arc in terms of *x*.

## Markscheme

area of triangle $= \frac{1}{2}(2x)^2 \sin \frac{\pi}{3}$ ***(M1)***

$= x^2\sqrt{3}$ ***A1***

**Note:** A $0.5 \times \text{base} \times \text{height}$ calculation is acceptable.

area of sector $= \frac{\theta}{2} r^2 = \frac{\pi}{6} r^2$ ***(M1)A1***

area of triangle is twice the area of the sector

$\Rightarrow 2\left(\frac{\pi}{6} r^2\right) = x^2\sqrt{3}$ ***M1***

$\Rightarrow r = x\sqrt{\frac{3\sqrt{3}}{\pi}}$ or equivalent ***A1***

***[6 marks]***

**10.** *[5 marks]*

The triangle ABC is equilateral of side 3 cm. The point D lies on [BC] such that BD = 1 cm.

Find $\cos \mathrm{D\hat{A}C}$.

## Markscheme

**METHOD 1**

$\mathrm{AD}^2 = 2^2 + 3^2 - 2 \times 2 \times 3 \times \cos 60^\circ$ ***M1***

(or $\mathrm{AD}^2 = 1^2 + 3^2 - 2 \times 1 \times 3 \times \cos 60^\circ$)

**Note:** ***M1*** for use of cosine rule with 60° angle.

$\mathrm{AD}^2 = 7$ ***A1***

$\cos \mathrm{D\hat{A}C} = \frac{9+7-4}{2\times3\times\sqrt{7}}$ ***M1A1***

**Note:** **M1** for use of cosine rule involving $\mathrm{D\hat{A}C}$.

$= \frac{2}{\sqrt{7}}$ ***A1***

**METHOD 2**

let point E be the foot of the perpendicular from D to AC

EC = 1 (by similar triangles, or triangle properties) ***M1A1***

(or AE = 2)

$\mathrm{DE} = \sqrt{3}$ and $\mathrm{AD} = \sqrt{7}$ (by Pythagoras) ***(M1)A1***

$\cos \mathrm{D\hat{A}C} = \frac{2}{\sqrt{7}}$ ***A1***

**Note:** If first ***M1*** not awarded but remainder of the question is correct award ***M0A0M1A1A1***.

***[5 marks]***

**11.** *[7 marks]*

The first three terms of a geometric sequence are $\sin x,\ \sin 2x$ and $4 \sin x \cos^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

(a) Find the common ratio *r*.

(b) Find the set of values of *x* for which the geometric series $\sin x + \sin 2x + 4 \sin x \cos^2 x + \ldots$ converges.

Consider $x = \arccos\left(\frac{1}{4}\right),\ x > 0$.

(c) Show that the sum to infinity of this series is $\frac{\sqrt{15}}{2}$.

## Markscheme

(a) $\sin x,\ \sin 2x$ and $4 \sin x \cos^2 x$

$r = \frac{2 \sin x \cos x}{\sin x} = 2 \cos x$ **A1**

**Note:** Accept $\frac{\sin 2x}{\sin x}$.

***[1 mark]***

(b) **EITHER**

$|r| < 1 \Rightarrow |2 \cos x| < 1$ ***M1***

**OR**

$-1 < r < 1 \Rightarrow -1 < 2 \cos x < 1$ ***M1***

**THEN**

$0 < \cos x < \frac{1}{2}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$-\frac{\pi}{2} < x < -\frac{\pi}{3}$ or $\frac{\pi}{3} < x < \frac{\pi}{2}$ ***A1A1***

***[3 marks]***

(c) $S_\infty = \frac{\sin x}{1-2\cos x}$ ***M1***

$$S_\infty = \frac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1-2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}$$

$$= \frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}$$ ***A1A1***

**Note:** Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{\sqrt{15}}{2}$ ***AG***

***[3 marks]***

***Total [7 marks]***

**12a.** *[2 marks]*

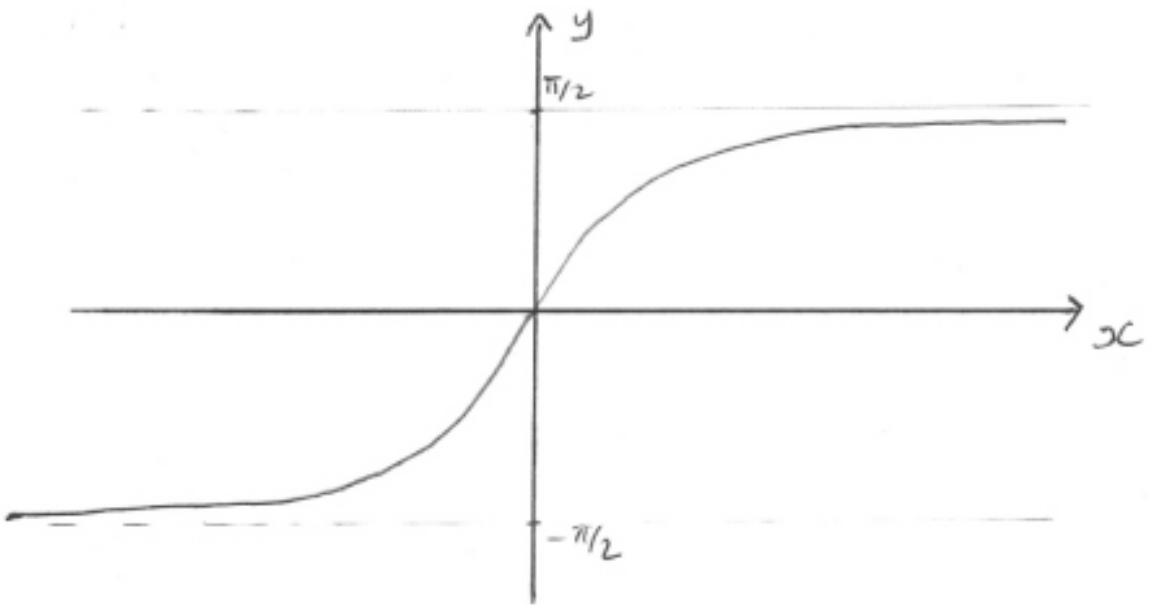
Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$

$g(x) = \frac{1}{x}, x \in \mathbb{R},\ x \neq 0$

Sketch the graph of $y = h(x)$.

Markscheme




***A1A1***

**Note:** ***A1*** for correct shape, ***A1*** for asymptotic behaviour at $y = \pm \frac{\pi}{2}$.

***[2 marks]***

**12b.** *[2 marks]*

Find an expression for the composite function $h \circ g(x)$ and state its domain.

Markscheme

$h \circ g(x) = \arctan\left(\frac{1}{x}\right)$ ***A1***

domain of $h \circ g$ is equal to the domain of $g : x \in \circ,\ x \neq 0$ ***A1***

***[2 marks]***

**12c.** *[7 marks]*

Given that $f(x) = h(x) + h \circ g(x)$,

(i) find $f'(x)$ in simplified form;

(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$.

Markscheme

(i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$ ***M1A1***

$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$ ***(A1)***

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$

$= 0$ ***A1***

(ii) **METHOD 1**

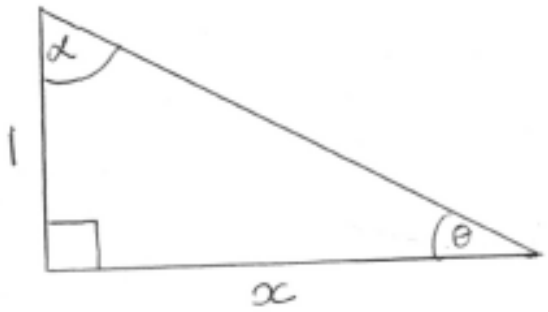
*f* is a constant ***R1***

when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$ ***M1A1***

$= \frac{\pi}{2}$ ***AG***

**METHOD 2**




from diagram

$\theta = \arctan\frac{1}{x}$ ***A1***

$\alpha = \arctan x$ ***A1***

$\theta + \alpha = \frac{\pi}{2}$ ***R1***

hence $f(x) = \frac{\pi}{2}$ ***AG***

**METHOD 3**

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ ***M1***

$= \frac{x+\frac{1}{x}}{1-x\left(\frac{1}{x}\right)}$ ***A1***

denominator = 0, so $f(x) = \frac{\pi}{2}$ $(\text{for } x > 0)$ ***R1***

***[7 marks]***

**12d.** *[3 marks]*

Nigel states that $f$ is an odd function and Tom argues that $f$ is an even function.

(i) State who is correct and justify your answer.

(ii) Hence find the value of $f(x)$ for $x < 0$.

Markscheme

(i) Nigel is correct. ***A1***

**METHOD 1**

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function ***R1***

**METHOD 2**

$$f(-x) = \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\arctan(x) - \arctan\left(\frac{1}{x}\right) = -f(x)$$

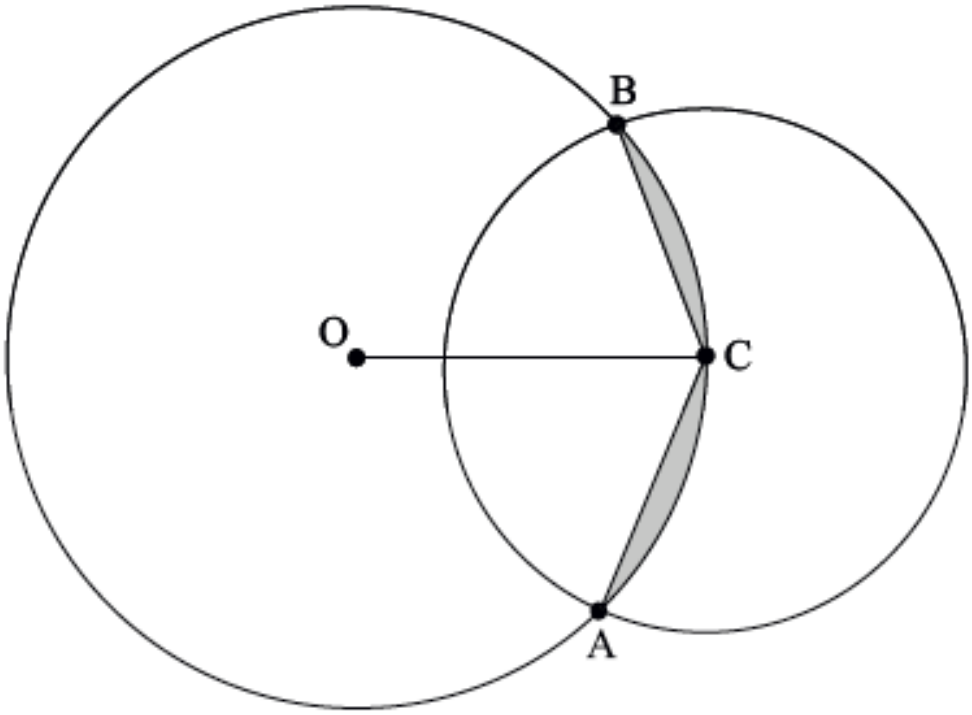
therefore *f* is an odd function. ***R1***

(ii) $f(x) = -\frac{\pi}{2}$ ***A1***

***[3 marks]***

**13.** *[6 marks]*

The following diagram shows two intersecting circles of radii 4 cm and 3 cm. The centre C of the smaller circle lies on the circumference of the bigger circle. O is the centre of the bigger circle and the two circles intersect at points A and B.




Find:

(a) $\mathrm{B\hat{O}C}$;

(b) the area of the shaded region.

## Markscheme

(a) **METHOD 1**

$2\arcsin\left(\frac{1.5}{4}\right)$ ***M1***

$\alpha = 0.769^c\ (44.0^\circ)$ ***A1***

**METHOD 2**

using the cosine rule:

$3^2 = 4^2 + 4^2 - 2(4)(4)\cos\alpha$ ***M1***

$\alpha = 0.769^c\ (44.0^\circ)$ ***A1***

***[2 marks]***

(b) one segment

$\mathrm{A}_1 = \frac{1}{2} \times 4^2 \times 0.76879 - \frac{1}{2} \times 4^2 \sin(0.76879)$ ***M1A1***

$= 0.58819\mathrm{K}$ ***(A1)***

$2\mathrm{A}_1 = 1.18\ (\mathrm{cm}^2)$ ***A1***

**Note:** Award ***M1*** only if both sector and triangle are considered.

***[4 marks]***

***Total [6 marks]***

**14.** *[8 marks]*

(a) Prove the trigonometric identity $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$.

(b) Given $f(x) = \sin(x + \frac{\pi}{6})\sin(x - \frac{\pi}{6}),\ x \in [0,\ \pi]$, find the range of $f$.

(c) Given $g(x) = \csc(x + \frac{\pi}{6})\csc(x - \frac{\pi}{6}),\ x \in [0,\ \pi],\ x \neq \frac{\pi}{6},\ x \neq \frac{5\pi}{6}$, find the range of $g$.

Markscheme

(a) $\sin(x+y)\sin(x-y)$

$= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$ ***M1A1***

$= \sin^2 x \cos^2 y + \sin x \sin y \cos x \cos y - \sin x \sin y \cos x \cos y - \cos^2 x \sin^2 y$

$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$ ***A1***

$= \sin^2 x(1 - \sin^2 y) - \sin^2 y(1 - \sin^2 x)$ ***A1***

$= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y$

$= \sin^2 x - \sin^2 y$ ***AG***

***[4 marks]***

(b) $f(x) = \sin^2 x - \frac{1}{4}$

range is $f \in \left[-\frac{1}{4},\ \frac{3}{4}\right]$ ***A1A1***

**Note:** Award ***A1*** for each end point. Condone incorrect brackets.

***[2 marks]***

(c) $g(x) = \frac{1}{\sin^2 x - \frac{1}{4}}$

range is $g \in ]-\infty,\ -4] \cup \left[\frac{4}{3}\ ,\ \infty\right[$ ***A1A1***

**Note:** Award ***A1*** for each part of range. Condone incorrect brackets.

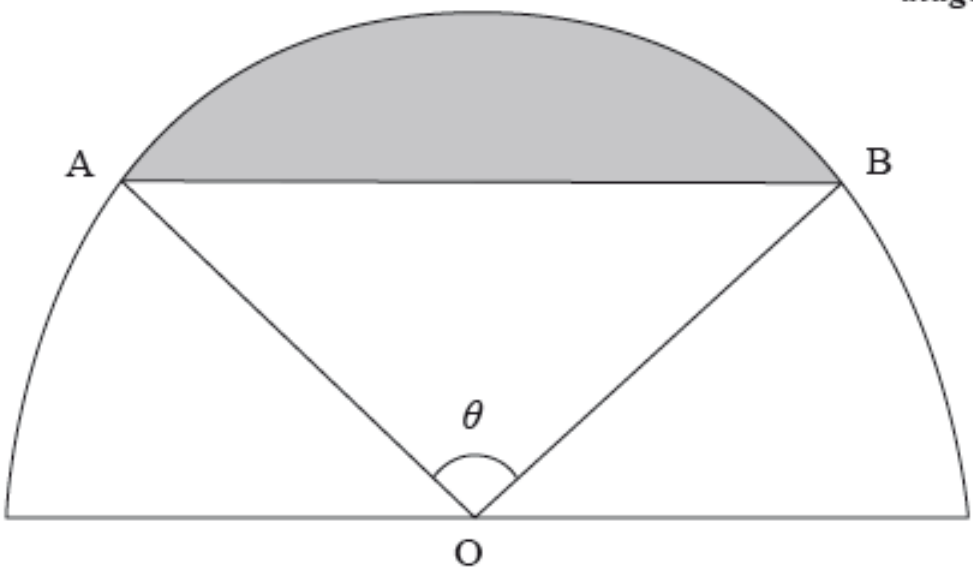
***[2 marks]***

***Total [8 marks]***

**15a.** *[2 marks]*

The diagram below shows a semi-circle of diameter 20 cm, centre O and two points A and B such that $\text{A}\hat{\text{O}}\text{B} = \theta$, where $\theta$ is in radians.

***diagram not to scale***




Show that the shaded area can be expressed as $50\theta - 50\sin\theta$.

Markscheme

$A = \frac{1}{2} \times 10^2 \times \theta - \frac{1}{2} \times 10^2 \times \sin\theta$ ***M1A1***

**Note:** Award ***M1*** for use of area of segment = area of sector – area of triangle.

$= 50\theta - 50\sin\theta$ ***AG***

***[2 marks]***

**15b.** *[3 marks]*

Find the value of $\theta$ for which the shaded area is equal to half that of the unshaded area, giving your answer correct to four significant figures.

## Markscheme

**METHOD 1**

unshaded area $= \frac{\pi \times 10^2}{2} - 50(\theta - \sin\theta)$

(or equivalent *eg* $50\pi - 50\theta + 50\sin\theta$) ***(M1)***

$50\theta - 50\sin\theta = \frac{1}{2}(50\pi - 50\theta + 50\sin\theta)$ ***(A1)***

$3\theta - 3\sin\theta - \pi = 0$

$\Rightarrow \theta = 1.969\ (\text{rad})$ ***A1***

**METHOD 2**

$50\theta - 50\sin\theta = \frac{1}{3}\left(\frac{\pi \times 10^2}{2}\right)$ ***(M1)(A1)***

$3\theta - 3\sin\theta - \pi = 0$

$\Rightarrow \theta = 1.969\ (\text{rad})$ ***A1***

***[3 marks]***

**16.** *[5 marks]*

A circular disc is cut into twelve sectors whose areas are in an arithmetic sequence.

The angle of the largest sector is twice the angle of the smallest sector.

Find the size of the angle of the smallest sector.

## Markscheme

**METHOD 1**

If the areas are in arithmetic sequence, then so are the angles. ***(M1)***

$\Rightarrow S_n = \frac{n}{2}(a + l) \Rightarrow \frac{12}{2}(\theta + 2\theta) = 18\theta$ ***M1A1***

$\Rightarrow 18\theta = 2\pi$ ***(A1)***

$\theta = \frac{\pi}{9}$ (accept $20°$) ***A1***

***[5 marks]***

**METHOD 2**

$a_{12} = 2a_1$ ***(M1)***

$\frac{12}{2}(a_1 + 2a_1) = \pi r^2$ ***M1A1***

$3a_1 = \frac{\pi r^2}{6}$

$\frac{3}{2} r^2\theta = \frac{\pi r^2}{6}$ ***(A1)***

$\theta = \frac{2\pi}{18} = \frac{\pi}{9}$ (accept $20°$) ***A1***

***[5 marks]***

**METHOD 3**

Let smallest angle = *a* , common difference = *d*

$a + 11d = 2a$ ***(M1)***

$a = 11d$ ***A1***

$S_n = \frac{12}{2}(2a + 11d) = 2\pi$ ***M1***

$6(2a + a) = 2\pi$ ***(A1)***

$18a = 2\pi$

$a = \frac{\pi}{9}$ (accept $20°$) ***A1***

***[5 marks]***

**17.** *[5 marks]*

In triangle ABC, AB = 9 cm , AC = 12 cm , and $\hat{B}$ is twice the size of $\hat{C}$ .

Find the cosine of $\hat{C}$ .

Markscheme

$\frac{9}{\sin C} = \frac{12}{\sin B}$ ***(M1)***

$\frac{9}{\sin C} = \frac{12}{\sin 2C}$ ***A1***

Using double angle formula $\frac{9}{\sin C} = \frac{12}{2\sin C \cos C}$ ***M1***

$\Rightarrow 9(2\sin C \cos C) = 12\sin C$

$\Rightarrow 6\sin C(3\cos C - 2) = 0$ or equivalent ***(A1)***

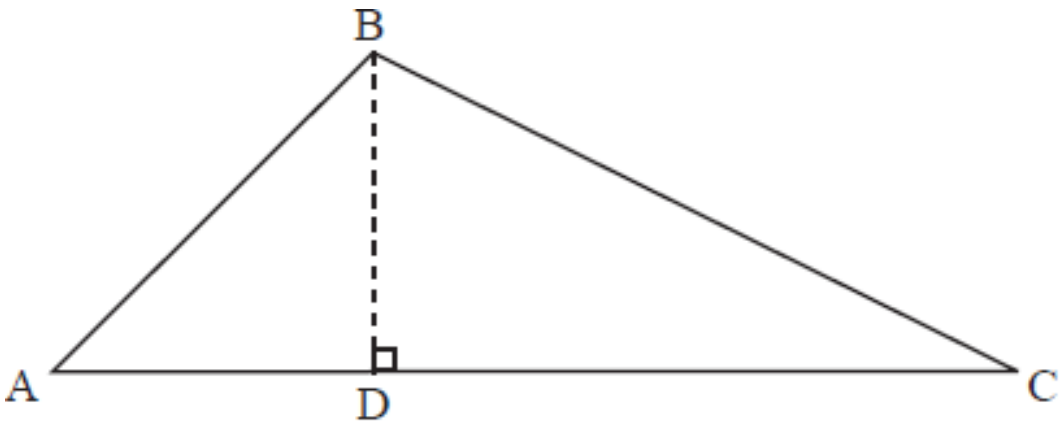
$(\sin C \neq 0)$

$\Rightarrow \cos C = \frac{2}{3}$ ***A1***

***[5 marks]***

**18.** *[12 marks]*

In triangle ABC, BC = *a* , AC = *b* , AB = *c* and [BD] is perpendicular to [AC].




(a) Show that $\mathrm{CD} = b - c\cos A$.

(b) **Hence**, by using Pythagoras' Theorem in the triangle BCD, prove the cosine rule for the triangle ABC.

(c) If $\mathrm{A\hat{B}C} = 60°$ , use the cosine rule to show that $c = \frac{1}{2}a \pm \sqrt{b^2 - \frac{3}{4}a^2}$ .

Markscheme

(a) $\mathrm{CD} = \mathrm{AC} - \mathrm{AD} = b - c\cos A$ ***R1AG***

***[1 mark]***

(b) **METHOD 1**

$\mathrm{BC}^2 = \mathrm{BD}^2 + \mathrm{CD}^2$ ***(M1)***

$a^2 = (c\sin A)^2 + (b - c\cos A)^2$ ***(A1)***

$= c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A$ ***A1***

$= b^2 + c^2 - 2bc\cos A$ ***A1***

***[4 marks]***

**METHOD 2**

$\mathrm{BD}^2 = \mathrm{AB}^2 - \mathrm{AD}^2 = \mathrm{BC}^2 - \mathrm{CD}^2$ ***(M1)(A1)***

$\Rightarrow c^2 - c^2\cos^2 A = a^2 - b^2 + 2bc\cos A - c^2\cos^2 A$ ***A1***

$\Rightarrow a^2 = b^2 + c^2 - 2bc\cos A$ ***A1***

***[4 marks]***

(c) **METHOD 1**

$b^2 = a^2 + c^2 - 2ac\cos 60^\circ \Rightarrow b^2 = a^2 + c^2 - ac$ ***(M1)A1***

$\Rightarrow c^2 - ac + a^2 - b^2 = 0$ ***M1***

$\Rightarrow c = \frac{a \pm \sqrt{(-a)^2 - 4(a^2 - b^2)}}{2}$ ***(M1)A1***

$= \frac{a \pm \sqrt{4b^2 - 3a^2}}{2} = \frac{a}{2} \pm \sqrt{\frac{4b^2 - 3a^2}{4}}$ ***(M1)A1***

$= \frac{1}{2}a \pm \sqrt{b^2 - \frac{3}{4}a^2}$ ***AG***

**Note:** Candidates can only obtain a maximum of the first three marks if they **verify** that the answer given in the question satisfies the equation.

***[7 marks]***

**METHOD 2**

$b^2 = a^2 + c^2 - 2ac\cos 60^\circ \Rightarrow b^2 = a^2 + c^2 - ac$ ***(M1)A1***

$c^2 - ac = b^2 - a^2$ ***(M1)***

$c^2 - ac + \left(\frac{a}{2}\right)^2 = b^2 - a^2 + \left(\frac{a}{2}\right)^2$ ***M1A1***

$\left(c - \frac{a}{2}\right)^2 = b^2 - \frac{3}{4}a^2$ ***(A1)***

$c - \frac{a}{2} = \pm\sqrt{b^2 - \frac{3}{4}a^2}$ ***A1***

$\Rightarrow c = \frac{1}{2}a \pm \sqrt{b^2 - \frac{3}{4}a^2}$ ***AG***

***[7 marks]***

**19.** *[6 marks]*

The depth, $h(t)$ metres, of water at the entrance to a harbour at $t$ hours after midnight on a particular day is given by

$$h(t) = 8 + 4\sin\left(\frac{\pi t}{6}\right),\ 0 \leqslant t \leqslant 24.$$

(a) Find the maximum depth and the minimum depth of the water.

(b) Find the values of $t$ for which $h(t) \geqslant 8$.

Markscheme

(a) Either finding depths graphically, using $\sin\frac{\pi t}{6} = \pm 1$ or solving $h'(t) = 0$ for $t$ ***(M1)***

$h(t)_{\max} = 12\ (\text{m}),\ h(t)_{\min} = 4\ (\text{m})$ ***A1A1 N3***

(b) Attempting to solve $8 + 4\sin\frac{\pi t}{6} = 8$ algebraically or graphically ***(M1)***

$t \in [0, 6] \cup [12, 18] \cup \{24\}$ ***A1A1N3***

***[6 marks]***

**20.** *[7 marks]*

Consider triangle ABC with $\hat{\text{BAC}} = 37.8°$ , AB = 8.75 and BC = 6 .

Find AC.

## Markscheme

**METHOD 1**

Attempting to use the cosine rule *i.e.* $\mathrm{BC}^2 = \mathrm{AB}^2 + \mathrm{AC}^2 - 2 \times \mathrm{AB} \times \mathrm{AC} \times \cos \mathrm{B\hat{A}C}$ ***(M1)***

$6^2 = 8.75^2 + \mathrm{AC}^2 - 2 \times 8.75 \times \mathrm{AC} \times \cos 37.8^\circ$ (or equivalent) ***A1***

Attempting to solve the quadratic in AC *e.g.* graphically, numerically or with quadratic formula ***M1A1***

Evidence from a sketch graph or their quadratic formula (AC = …) that there are two values of AC to determine. ***(A1)***

AC = 9.60 or AC = 4.22 ***A1A1 N4***

**Note:** Award ***(M1)A1M1A1(A0)A1A0*** for one correct value of AC.

***[7 marks]***

**METHOD 2**

Attempting to use the sine rule *i.e.* $\frac{\mathrm{BC}}{\sin \mathrm{B\hat{A}C}} = \frac{\mathrm{AB}}{\sin \mathrm{A\hat{C}B}}$ ***(M1)***

$\sin C = \frac{8.75 \sin 37.8^\circ}{6} \quad (= 0.8938\ldots)$ ***(A1)***

$C = 63.3576\ldots^\circ$ ***A1***

$C = 116.6423\ldots^\circ$ **and** $B = 78.842\ldots^\circ$ or $B = 25.5576\ldots^\circ$ ***A1***

**EITHER**

Attempting to solve $\frac{\mathrm{AC}}{\sin 78.842\ldots^\circ} = \frac{6}{\sin 37.8^\circ}$ **or** $\frac{\mathrm{AC}}{\sin 25.5576\ldots^\circ} = \frac{6}{\sin 37.8^\circ}$ ***M1***

**OR**

Attempting to solve $\mathrm{AC}^2 = 8.75^2 + 6^2 - 2 \times 8.75 \times 6 \times \cos 25.5576\ldots^\circ$ **or**

$\mathrm{AC}^2 = 8.75^2 + 6^2 - 2 \times 8.75^2 \times 6 \times \cos 78.842\ldots^\circ$ ***M1***

$\mathrm{AC} = 9.60$ **or** $\mathrm{AC} = 4.22$ ***A1A1 N4***

**Note:** Award ***(M1)(A1)A1A0M1A1A0*** for one correct value of AC.

***[7 marks]***

**21.** *[7 marks]*

In a triangle ABC, $\hat{A} = 35^\circ$, BC = 4 cm and AC = 6.5 cm. Find the possible values of $\hat{B}$ and the corresponding values of AB.

Markscheme

$\frac{\sin B}{6.5} = \frac{\sin 35^\circ}{4}$ ***M1***

$\hat{B} = 68.8^\circ$ or $111^\circ$ ***A1A1***

$\hat{C} = 76.2^\circ$ or $33.8^\circ$ (accept $34^\circ$) ***A1***

$\frac{\text{AB}}{\sin C} = \frac{\text{BC}}{\sin A}$

$\frac{\text{AB}}{\sin 76.2^\circ} = \frac{4}{\sin 35^\circ}$ ***(M1)***

AB = 6.77 cm ***A1***

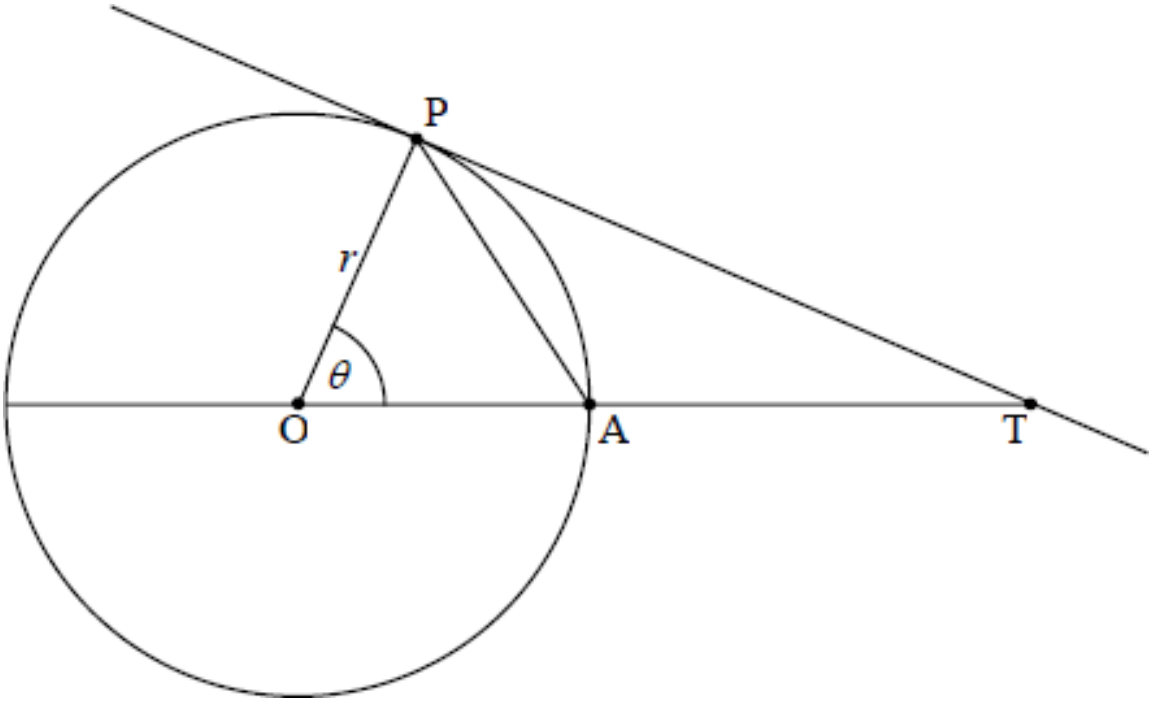
$\frac{\text{AB}}{\sin 33.8^\circ} = \frac{4}{\sin 35^\circ}$

AB = 3.88 cm (accept 3.90) ***A1***

***[7 marks]***

**22a.** *[1 mark]*

The diagram shows a tangent, (TP) , to the circle with centre O and radius *r* . The size of $\text{P}\hat{\text{O}}\text{A}$ is $\theta$ radians.




Find the area of triangle AOP in terms of $r$ and $\theta$.

Markscheme

area of $\text{AOP} = \frac{1}{2} r^2 \sin\theta$ ***A1***

***[1 mark]***

**22b.** *[2 marks]*

Find the area of triangle POT in terms of $r$ and $\theta$.

Markscheme

$\text{TP} = r\tan\theta$ ***(M1)***

area of POT $= \frac{1}{2} r(r\tan\theta)$

$= \frac{1}{2} r^2 \tan\theta$ ***A1***

***[2 marks]***

**22c.** *[2 marks]*

Using your results from part (a) and part (b), show that $\sin\theta < \theta < \tan\theta$.

Markscheme

area of sector OAP $= \frac{1}{2} r^2\theta$ ***A1***

area of triangle OAP < area of sector OAP < area of triangle POT ***R1***

$\frac{1}{2} r^2 \sin\theta < \frac{1}{2} r^2\theta < \frac{1}{2} r^2 \tan\theta$

$\sin\theta < \theta < \tan\theta$ ***AG***

***[2 marks]***

**23.** *[5 marks]*

The points P and Q lie on a circle, with centre O and radius 8 cm, such that $\text{P}\hat{\text{O}}\text{Q} = 59°$ .

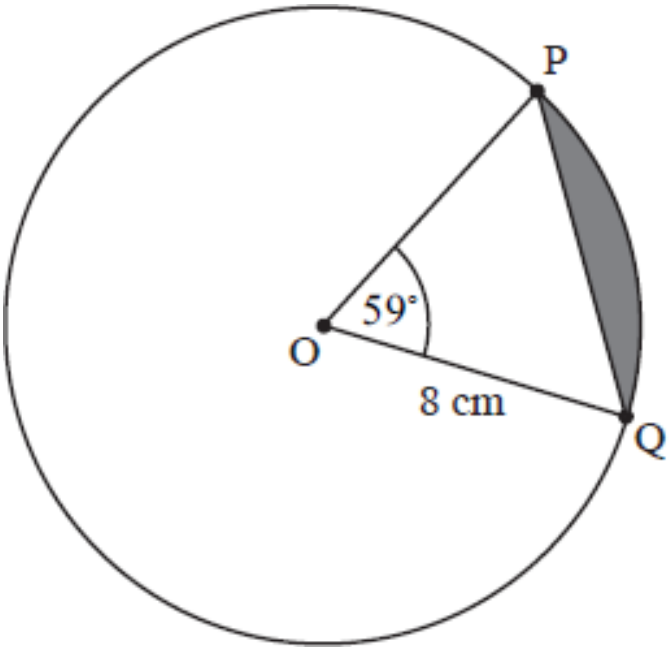



***diagram not to scale***

Find the area of the shaded segment of the circle contained between the arc PQ and the chord [PQ].

Markscheme

area of triangle $\text{POQ} = \frac{1}{2} 8^2 \sin 59°$ ***M1***

= 27.43 ***(A1)***

area of sector $= \pi 8^2 \frac{59}{360}$ ***M1***

= 32.95 ***(A1)***

area between arc and chord $= 32.95 - 27.43$

$= 5.52\ (\text{cm}^2)$ ***A1***

***[5 marks]***

**24.** *[6 marks]*

The vertices of an equilateral triangle, with perimeter *P* and area *A* , lie on a circle with radius *r* . Find an expression for $\frac{P}{A}$ in the form $\frac{k}{r}$, where $k \in \mathbb{Z}^+$.

## Markscheme

let the length of one side of the triangle be *x*

consider the triangle consisting of a side of the triangle and two radii

**EITHER**

$x^2 = r^2 + r^2 - 2r^2 \cos 120^\circ$ ***M1***

$= 3r^2$

**OR**

$x = 2r \cos 30^\circ$ ***M1***

**THEN**

$x = r\sqrt{3}$ ***A1***

so perimeter $= 3\sqrt{3}r$ ***A1***

now consider the area of the triangle

area $= 3 \times \frac{1}{2} r^2 \sin 120^\circ$ ***M1***

$= 3 \times \frac{\sqrt{3}}{4} r^2$ ***A1***

$\frac{P}{A} = \frac{3\sqrt{3}r}{\frac{3\sqrt{3}}{4}r^2}$

$= \frac{4}{r}$ ***A1***

**Note:** Accept alternative methods

***[6 marks]***

**25.** *[5 marks]*

(a) Show that $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

(b) Hence, or otherwise, find the value of $\arctan(2) + \arctan(3)$.

## Markscheme

(a) **METHOD 1**

let $x = \arctan\frac{1}{2} \Rightarrow \tan x = \frac{1}{2}$ and $y = \arctan\frac{1}{3} \Rightarrow \tan y = \frac{1}{3}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1$ ***M1***

so, $x + y = \arctan 1 = \frac{\pi}{4}$ ***A1AG***

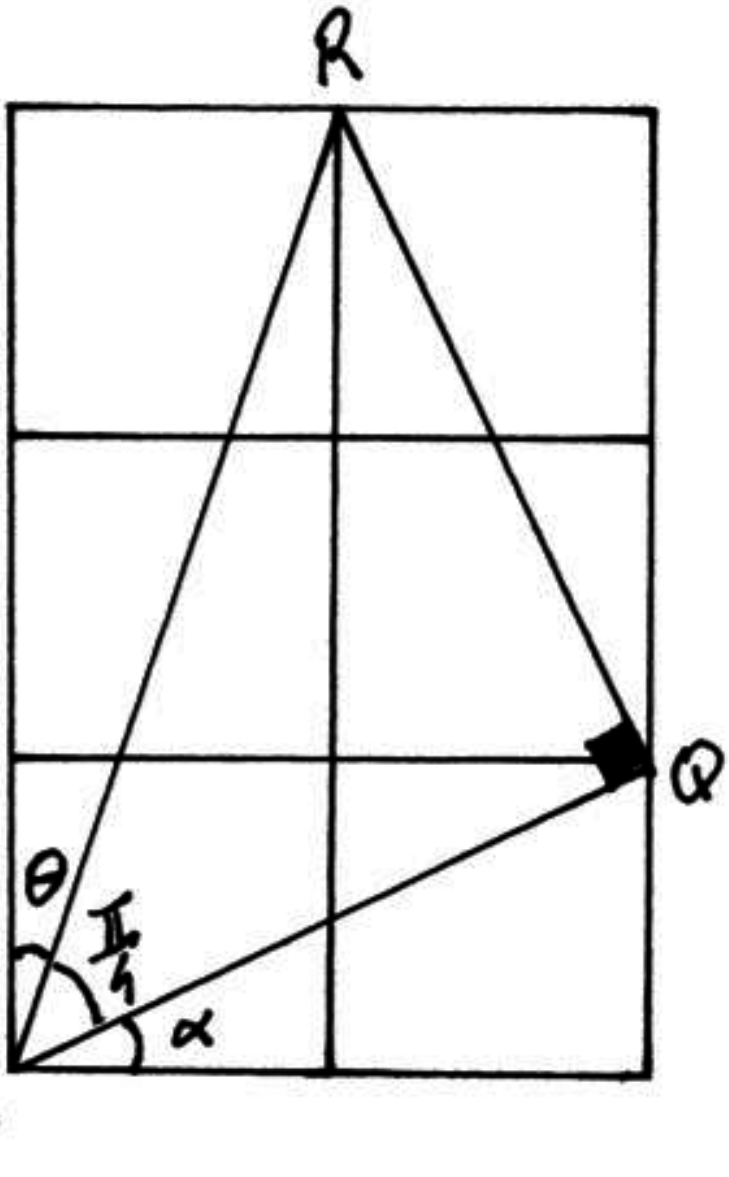
**METHOD 2**

for $x,\ y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$ ***M1***

so, $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right) = \frac{\pi}{4}$ ***A1AG***

**METHOD 3**

an appropriate sketch ***M1***




*e.g.*

correct reasoning leading to $\frac{\pi}{4}$ ***R1AG***

(b) **METHOD 1**

$\arctan(2) + \arctan(3) = \frac{\pi}{2} - \arctan\left(\frac{1}{2}\right) + \frac{\pi}{2} - \arctan\left(\frac{1}{3}\right)$ ***(M1)***

$= \pi - \left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right)$ ***(A1)***

**Note:** Only one of the previous two marks may be implied.

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ ***A1 N1***

**METHOD 2**

let $x = \arctan 2 \Rightarrow \tan x = 2$ and $y = \arctan 3 \Rightarrow \tan y = 3$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{2+3}{1-2\times 3} = -1$ ***(M1)***

as $\frac{\pi}{4} < x < \frac{\pi}{2} \quad \left(\text{accept } 0 < x < \frac{\pi}{2}\right)$

and $\frac{\pi}{4} < y < \frac{\pi}{2} \quad \left(\text{accept } 0 < y < \frac{\pi}{2}\right)$

$\frac{\pi}{2} < x + y < \pi \quad (\text{accept } 0 < x + y < \pi)$ ***(R1)***

**Note:** Only one of the previous two marks may be implied.

so, $x + y = \frac{3\pi}{4}$ ***A1 N1***

**METHOD 3**

for $x,\ y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi \text{ if } xy > 1$ ***(M1)***

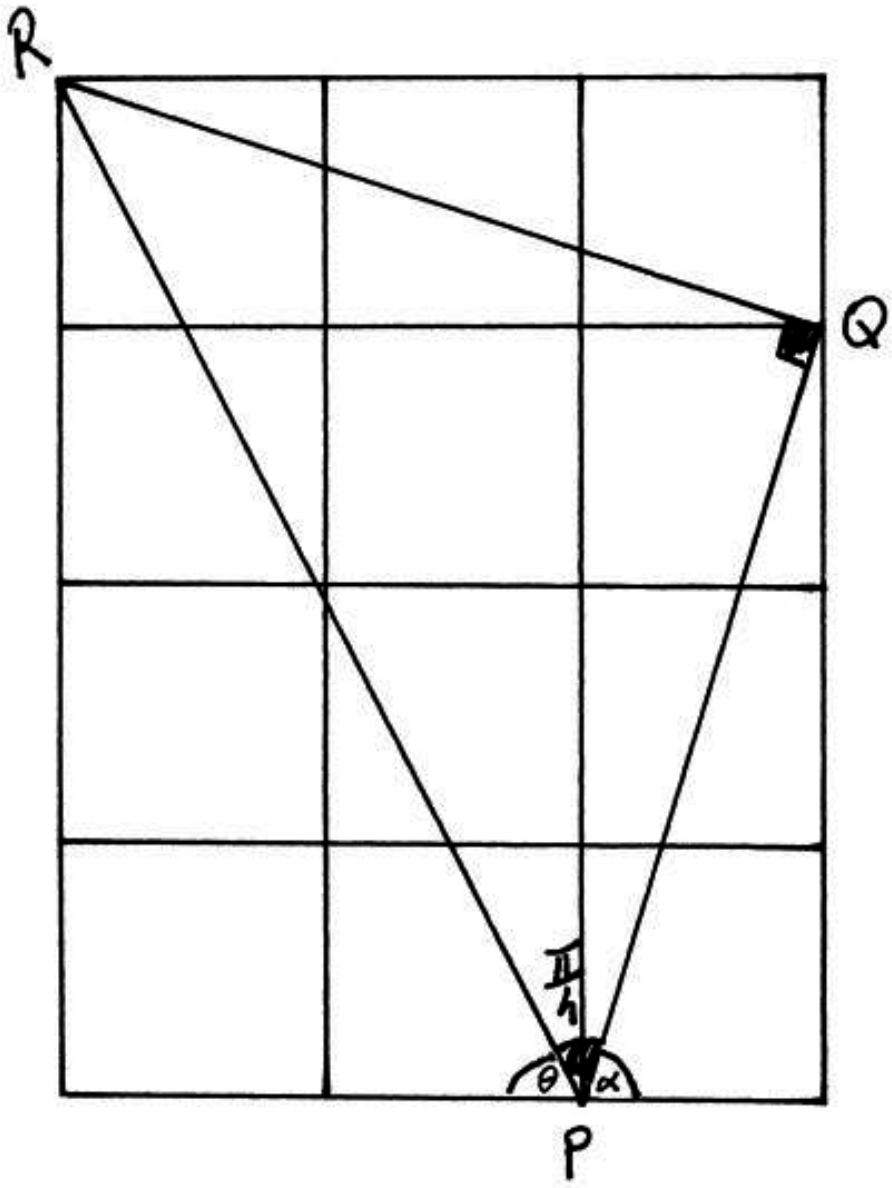
so, $\arctan 2 + \arctan 3 = \arctan\left(\frac{2+3}{1-2\times 3}\right) + \pi$ ***(A1)***

**Note:** Only one of the previous two marks may be implied.

$= \frac{3\pi}{4}$ ***A1 N1***

**METHOD 4**

an appropriate sketch ***M1***




*e.g.*

correct reasoning leading to $\frac{3\pi}{4}$ ***R1A1***

***[5 marks]***

**26.** *[5 marks]*

The diagram below shows two straight lines intersecting at O and two circles, each with centre O. The outer circle has radius *R* and the inner circle has radius *r* .

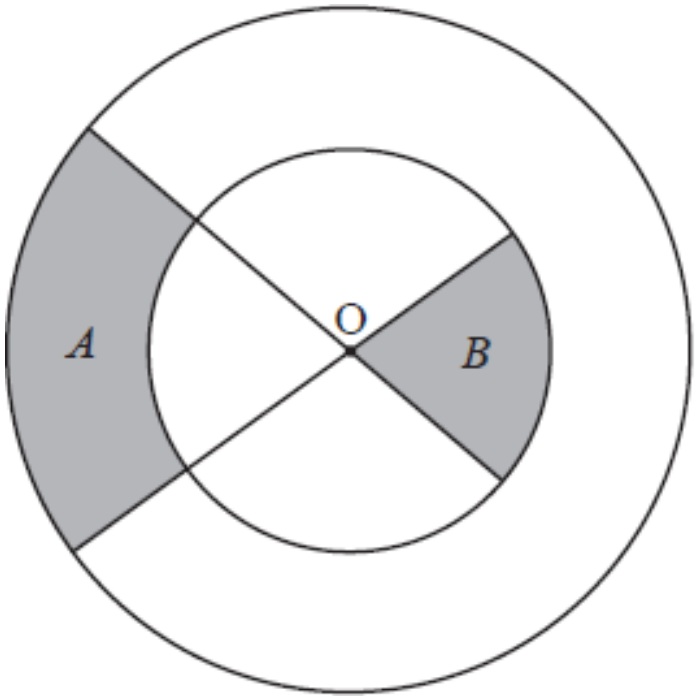



*diagram not to scale*

Consider the shaded regions with areas *A* and *B* . Given that $A : B = 2 : 1$, find the **exact** value of the ratio $R : r$ .

Markscheme

$A = \frac{\theta}{2}(R^2 - r^2)$ ***A1***

$B = \frac{\theta}{2} r^2$ ***A1***

from $A : B = 2 : 1$ , we have $R^2 - r^2 = 2r^2$ ***M1***

$R = \sqrt{3}r$ ***(A1)***

hence exact value of the ratio $R : r$ is $\sqrt{3} : 1$ ***A1 N0***

***[5 marks]***

**27.** *[8 marks]*

A triangle has sides of length $(n^2 + n + 1)$, $(2n + 1)$ and $(n^2 - 1)$ where $n > 1$.

(a) Explain why the side $(n^2 + n + 1)$ must be the longest side of the triangle.

(b) Show that the largest angle, $\theta$, of the triangle is $120^\circ$ .

Markscheme

(a) a reasonable attempt to show either that $n^2 + n + 1 > 2n + 1$ or

$n^2 + n + 1 > n^2 - 1$ ***M1***

complete solution to each inequality ***A1A1***

(b) $\cos\theta = \frac{(2n+1)^2+(n^2-1)^2-(n^2+n+1)^2}{2(2n+1)(n^2-1)}$ ***M1A1***

$= \frac{-2n^3-n^2+2n+1}{2(2n+1)(n^2-1)}$ ***M1***

$= -\frac{(n-1)(n+1)(2n+1)}{2(2n+1)(n^2-1)}$ ***A1***

$= -\frac{1}{2}$ ***A1***

$\theta = 120^\circ$ ***AG***

***[8 marks]***

**28a.** *[1 mark]*

The angle $\theta$ lies in the first quadrant and $\cos\theta = \frac{1}{3}$.

Write down the value of $\sin\theta$.

Markscheme

$\sin\theta = \frac{\sqrt{8}}{3}$ ***A1***

***[1 mark]***

**28b.** *[2 marks]*

Find the value of $\tan 2\theta$.

Markscheme

$\tan 2\theta = \frac{2\times\sqrt{8}}{1-8} = -\frac{2\sqrt{8}}{7} \quad \left(-\frac{4\sqrt{2}}{7}\right)$ ***M1A1***

***[2 marks]***

**28c.** *[3 marks]*

Find the value of $\cos\left(\frac{\theta}{2}\right)$, giving your answer in the form $\frac{\sqrt{a}}{b}$ where $a$, $b \in \mathbb{Z}^+$.

Markscheme

$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\frac{1}{3}}{2} = \frac{2}{3}$ ***M1A1***

$\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{6}}{3}$ ***A1***

***[3 marks]***

**29a.** *[3 marks]*

In the triangle ABC, $\mathrm{AB} = 2\sqrt{3}$, AC = 9 and $\mathrm{B\hat{A}C} = 150°$.

Determine BC, giving your answer in the form $k\sqrt{3}, k \in \mathbb{Z}^+$.

Markscheme

$\mathrm{BC}^2 = 12 + 81 + 2 \times 2\sqrt{3} \times 9 \times \frac{\sqrt{3}}{2} = 147$ ***M1A1***

$\mathrm{BC} = 7\sqrt{3}$ ***A1***

***[3 marks]***

**29b.** *[4 marks]*

The point D lies on (BC), and (AD) is perpendicular to (BC). Determine AD.

Markscheme

area of triangle $\mathrm{ABC} = \frac{1}{2} \times 9 \times 2\sqrt{3} \times \frac{1}{2} \left(= \frac{9\sqrt{3}}{2}\right)$ ***M1A1***

therefore $\frac{1}{2} \times \mathrm{AD} \times 7\sqrt{3} = \frac{9\sqrt{3}}{2}$ ***M1***

$\mathrm{AD} = \frac{9}{7}$ ***A1***

***[4 marks]***

**30.** *[6 marks]*

In the right circular cone below, O is the centre of the base which has radius 6 cm. The points B and C are on the circumference of the base of the cone. The height AO of the cone is 8 cm and the angle $\mathrm{B\hat{O}C}$ is 60°.

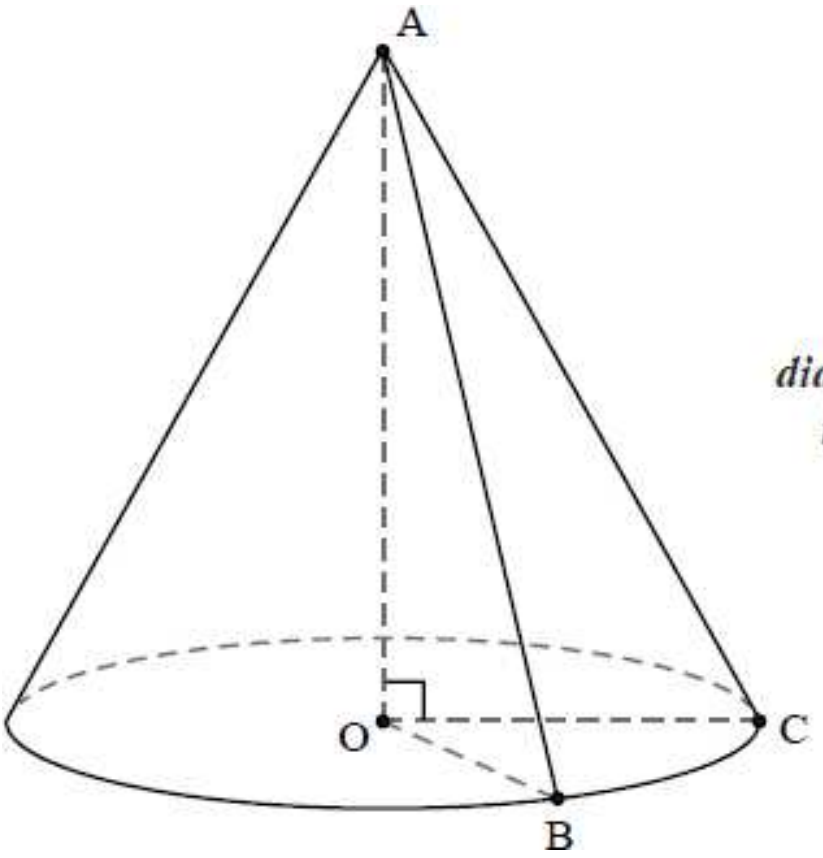



*diagram not to scale*

Calculate the size of the angle $\mathrm{B\hat{A}C}$.

## Markscheme

AC = AB = 10 (cm) ***A1***

triangle OBC is equilateral ***(M1)***

BC = 6 (cm) ***A1***

**EITHER**

$\mathrm{B\hat{A}C} = 2\arcsin\frac{3}{10}$ ***M1A1***

$\mathrm{B\hat{A}C} = 34.9°$ (accept 0.609 radians) ***A1***

**OR**

$\cos \mathrm{B\hat{A}C} = \frac{10^2+10^2-6^2}{2\times10\times10} = \frac{164}{200}$ ***M1A1***

$\mathrm{B\hat{A}C} = 34.9°$ (accept 0.609 radians) ***A1***

**Note:** Other valid methods may be seen.

***[6 marks]***

**31.** *[6 marks]*

Consider the triangle ABC where $\mathrm{B\hat{A}C} = 70°$, AB = 8 cm and AC = 7 cm. The point D on the side BC is such that $\frac{\mathrm{BD}}{\mathrm{DC}} = 2$.

Determine the length of AD.

Markscheme

use of cosine rule: $\mathrm{BC} = \sqrt{(8^2 + 7^2 - 2 \times 7 \times 8\cos 70)} = 8.6426\ldots$ ***(M1)A1***

**Note:** Accept an expression for $\mathrm{BC}^2$.

$\mathrm{BD} = 5.7617\ldots \quad (\mathrm{CD} = 2.88085\ldots)$ ***A1***

use of sine rule: $\hat{B} = \arcsin\left(\frac{7\sin 70}{\mathrm{BC}}\right) = 49.561\ldots° \quad (\hat{C} = 60.4387\ldots°)$ ***(M1)A1***

use of cosine rule: $\mathrm{AD} = \sqrt{8^2 + \mathrm{BD}^2 - 2 \times \mathrm{BD} \times 8\cos B} = 6.12\ (\mathrm{cm})$ ***A1***

**Note:** Scale drawing method not acceptable.

***[6 marks]***

**32.** *[12 marks]*

The interior of a circle of radius 2 cm is divided into an infinite number of sectors. The areas of these sectors form a geometric sequence with common ratio *k*. The angle of the first sector is $\theta$ radians.

(a) Show that $\theta = 2\pi(1-k)$.

(b) The perimeter of the third sector is half the perimeter of the first sector.

Find the value of *k* and of $\theta$.

Markscheme

(a) the area of the first sector is $\frac{1}{2}\,2^2\theta$ ***(A1)***

the sequence of areas is $2\theta,\ 2k\theta,\ 2k^2\theta \ldots$ ***(A1)***

the sum of these areas is $2\theta(1 + k + k^2 + \ldots)$ ***(M1)***

$= \frac{2\theta}{1-k} = 4\pi$ ***M1A1***

hence $\theta = 2\pi(1 - k)$ ***AG***

**Note:** Accept solutions where candidates deal with angles instead of area.

***[5 marks]***

(b) the perimeter of the first sector is $4 + 2\theta$ ***(A1)***

the perimeter of the third sector is $4 + 2k^2\theta$ ***(A1)***

the given condition is $4 + 2k^2\theta = 2 + \theta$ ***M1***

which simplifies to $2 = \theta(1 - 2k^2)$ ***A1***

eliminating $\theta$, obtain cubic in *k*: $\pi(1 - k)(1 - 2k^2) - 1 = 0$ ***A1***

or equivalent

solve for *k* = 0.456 and then $\theta = 3.42$ ***A1A1***

***[7 marks]***

***Total [12 marks]***

**33.** *[6 marks]*

Triangle ABC has AB = 5 cm, BC = 6 cm and area 10 $\text{cm}^2$.

(a) Find $\sin \hat{B}$.

(b) **Hence**, find the two possible values of AC, giving your answers correct to two decimal places.

Markscheme

(a) area $= \frac{1}{2} \times \text{BC} \times \text{AB} \times \sin B$ ***(M1)***

$\left(10 = \frac{1}{2} \times 5 \times 6 \times \sin B\right)$

$\sin \hat{B} = \frac{2}{3}$ ***A1***

(b) $\cos B = \pm \frac{\sqrt{5}}{3}$ $(= \pm 0.7453\ldots)$ or $B = 41.8\ldots$ and $138.1\ldots$ ***(A1)***

$\mathrm{AC}^2 = \mathrm{BC}^2 + \mathrm{AB}^2 - 2 \times \mathrm{BC} \times \mathrm{AB} \times \cos B$ ***(M1)***

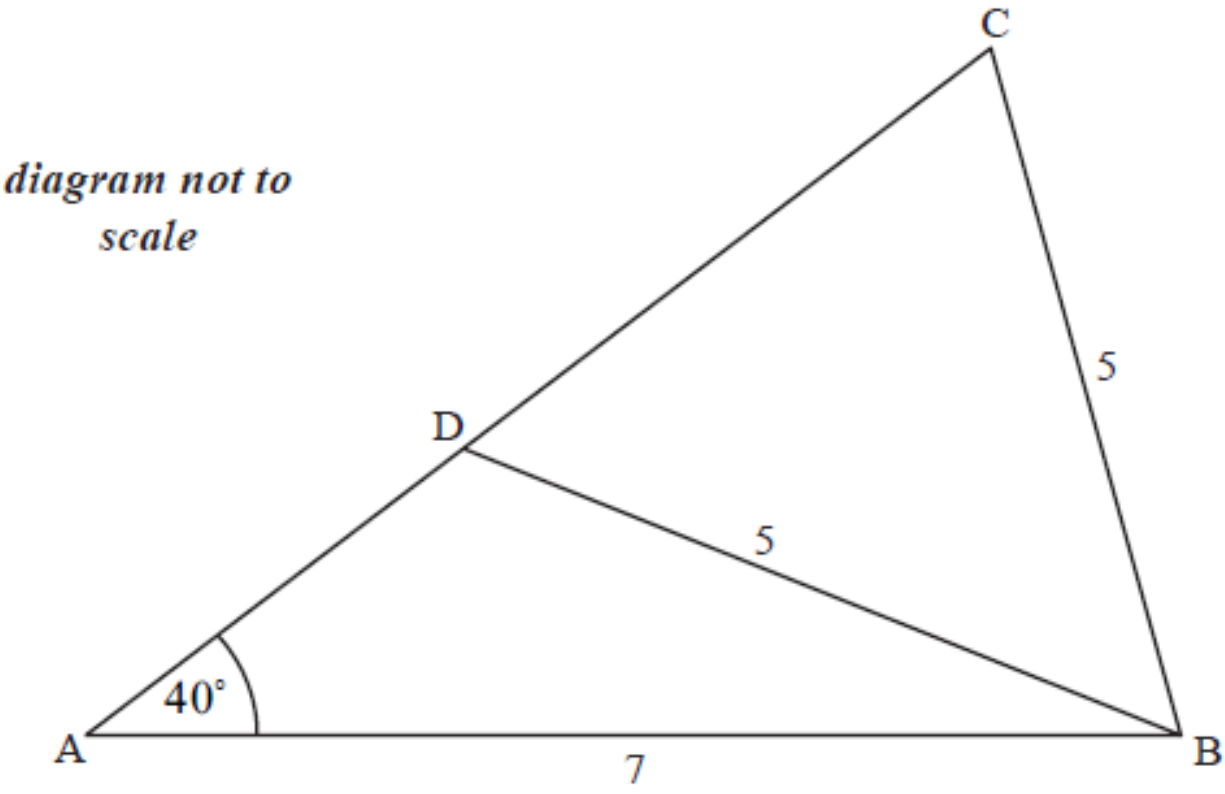
$\mathrm{AC} = \sqrt{5^2 + 6^2 - 2 \times 5 \times 6 \times 0.7453\ldots}$ or $\sqrt{5^2 + 6^2 + 2 \times 5 \times 6 \times 0.7453\ldots}$

$\mathrm{AC} = 4.03$ or $10.28$ ***A1A1***

***[6 marks]***

**34.** *[5 marks]*

Given $\Delta$ABC, with lengths shown in the diagram below, find the length of the line segment [CD].




Markscheme

**METHOD 1**

$\frac{\sin C}{7} = \frac{\sin 40}{5}$ ***M1(A1)***

$\hat{\mathrm{BCD}} = 64.14\ldots^\circ$ ***A1***

$\text{CD} = 2 \times 5 \cos 64.14\ldots$ ***M1***

**Note:** Also allow use of sine or cosine rule.

$\text{CD} = 4.36$ ***A1***

***[5 marks]***

**METHOD 2**

let $\text{AC} = x$

cosine rule

$5^2 = 7^2 + x^2 - 2 \times 7 \times x \cos 40$ ***M1A1***

$x^2 - 10.7\ldots x + 24 = 0$

$x = \frac{10.7\ldots \pm \sqrt{(10.7\ldots)^2 - 4 \times 24}}{2}$ ***(M1)***

$x = 7.54; 3.18$ ***(A1)***

CD is the difference in these two values $= 4.36$ ***A1***

**Note:** Other methods may be seen.

***[5 marks]***

**35.** *[7 marks]*

The radius of the circle with centre C is 7 cm and the radius of the circle with centre D is 5 cm. If the length of the chord [AB] is 9 cm, find the area of the shaded region enclosed by the two arcs AB.

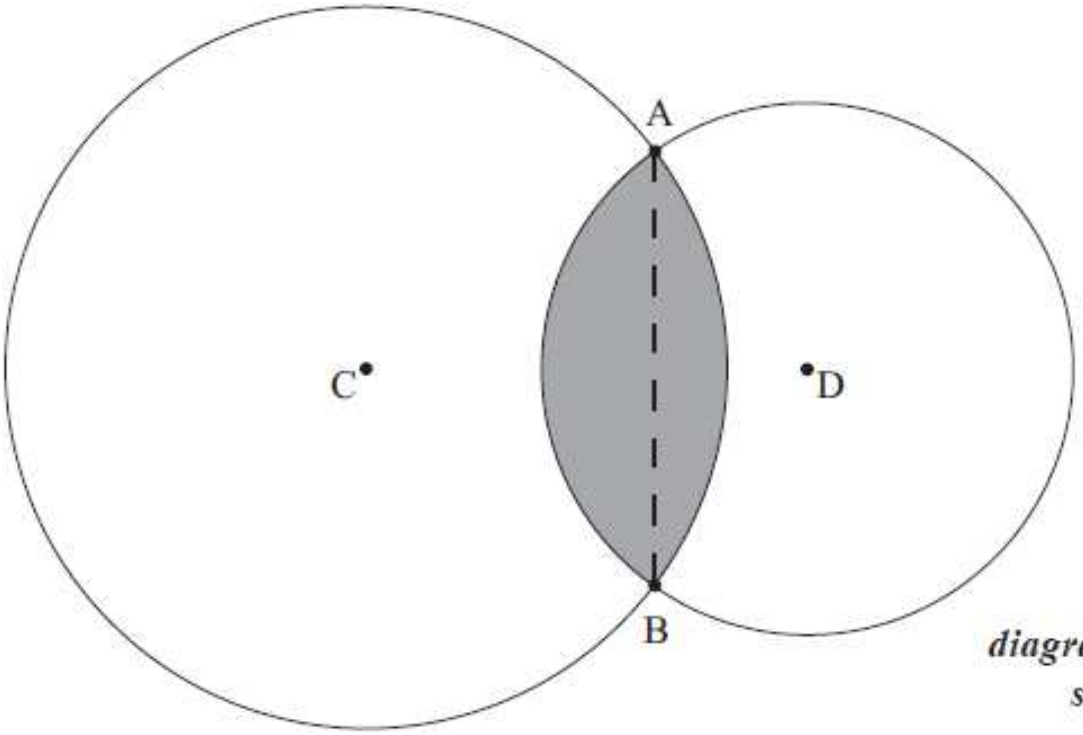
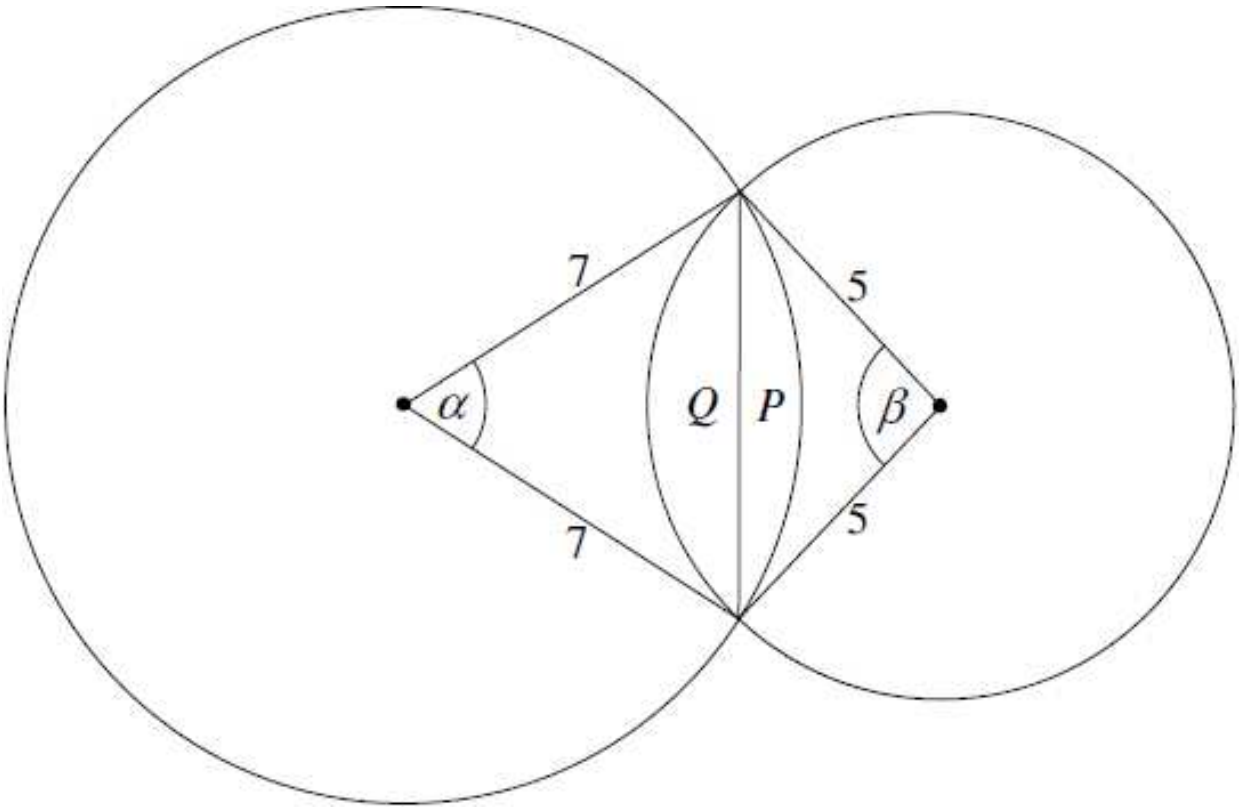



*diagram not to scale*

Markscheme




$\alpha = 2\arcsin\left(\frac{4.5}{7}\right)$ $(\Rightarrow \alpha = 1.396... = 80.010°...)$ ***M1(A1)***

$\beta = 2\arcsin\left(\frac{4.5}{5}\right)$ $(\Rightarrow \beta = 2.239\ldots = 128.31^\circ\ldots)$ ***(A1)***

**Note:** Allow use of cosine rule.

area $P = \frac{1}{2} \times 7^2 \times (\alpha - \sin\alpha) = 10.08\ldots$ **M1(A1)**

area $Q = \frac{1}{2} \times 5^2 \times (\beta - \sin\beta) = 18.18\ldots$ ***(A1)***

**Note:** The ***M1*** is for an attempt at area of sector minus area of triangle.

**Note:** The use of degrees correctly converted is acceptable.

area = 28.3 (cm) ***A1***

***[7 marks]***

**36.** *[7 marks]*

The diagram below shows two concentric circles with centre O and radii 2 cm and 4 cm.

The points P and Q lie on the larger circle and $\text{P}\hat{\text{O}}\text{Q} = x$, where $0 < x < \frac{\pi}{2}$.

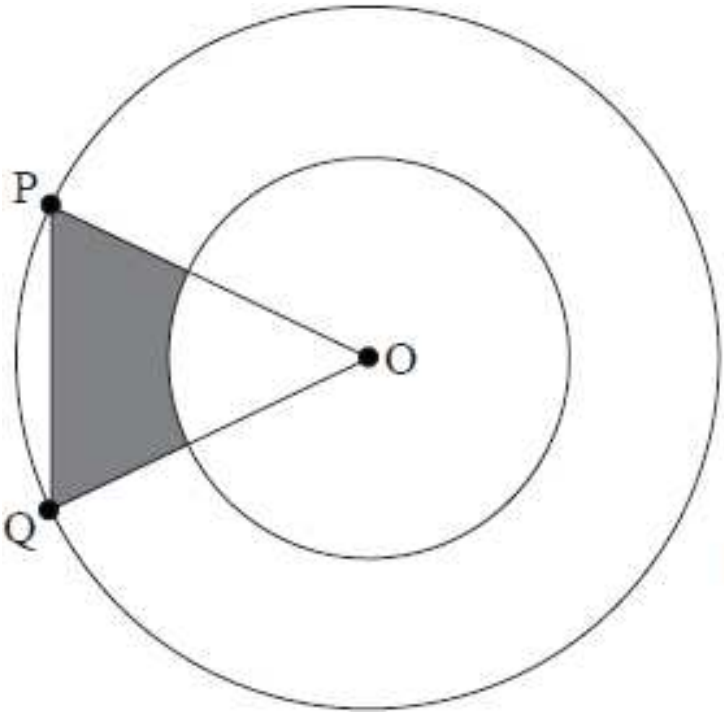



*diagram not to scale*

(a) Show that the area of the shaded region is $8\sin x - 2x$.

(b) Find the maximum area of the shaded region.

## Markscheme

(a) shaded area area of triangle area of sector, *i.e.* ***(M1)***

$\left(\frac{1}{2} \times 4^2 \sin x\right) - \left(\frac{1}{2}\, 2^2 x\right) = 8 \sin x - 2x$ ***A1A1AG***

(b) **EITHER**

any method from GDC gaining $x \approx 1.32$ ***(M1)(A1)***

maximum value for given domain is $5.11$ ***A2***

**OR**

$\frac{\mathrm{d}A}{\mathrm{d}x} = 8 \cos x - 2$ ***A1***

set $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$, hence $8 \cos x - 2 = 0$ ***M1***

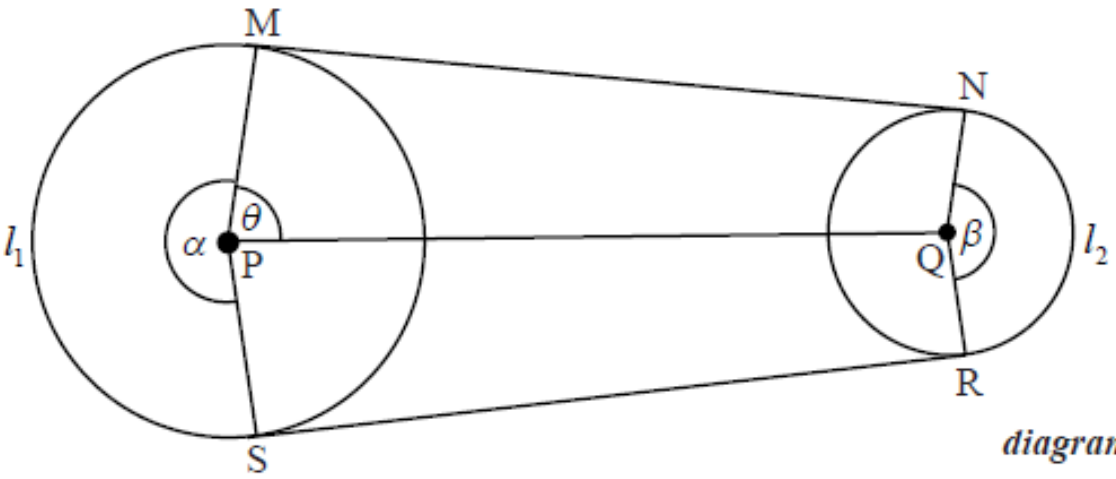
$\cos x = \frac{1}{4} \Rightarrow x \approx 1.32$ ***A1***

hence $A_{\max} = 5.11$ ***A1***

***[7 marks]***

**37.** *[18 marks]*

Two non-intersecting circles C , containing points M and S , and C , containing points N and R, have centres P and Q where PQ $= 50$ . The line segments [MN] and [SR] are common tangents to the circles. The size of the reflex angle MPS is $\alpha$, the size of the obtuse angle NQR is $\beta$ , and the size of the angle MPQ is $\theta$ . The arc length MS is $l_1$ and the arc length NR is $l_2$ . This information is represented in the diagram below.




The radius of C is $x$, where $x \geqslant 10$ and the radius of C is $10$.

(a) Explain why $x < 40$.

(b) Show that cosθ = x −10 50.

(c) (i) Find an expression for MN in terms of $x$.

(ii) Find the value of $x$ that maximises MN.

(d) Find an expression in terms of $x$ for

(i) $\alpha$;

(ii) $\beta$.

(e) The length of the perimeter is given by $l_1 + l_2 + \text{MN} + \text{SR}$.

(i) Find an expression, $b(x)$, for the length of the perimeter in terms of $x$.

(ii) Find the maximum value of the length of the perimeter.

(iii) Find the value of $x$ that gives a perimeter of length $200$.

## Markscheme

(a) PQ $= 50$ and non-intersecting ***R1***

***[1 mark]***

(b) a construction QT (where T is on the radius MP), parallel to MN, so that $\text{Q}\hat{\text{T}}\text{M} = 90^\circ$ (angle between tangent and radius $= 90^\circ$ ) ***M1***

lengths $50, x-10$ and angle $\theta$ marked on a diagram, or equivalent ***R1***

**Note:** Other construction lines are possible.

***[2 marks]***

(c) (i) MN $= \sqrt{50^2-(x-10)^2}$ ***A1***

(ii) maximum for MN occurs when $x = 10$ ***A1***

***[2 marks]***

(d) (i) $\alpha = 2\pi - 2\theta$ ***M1***

$= 2\pi - 2\arccos\left(\frac{x-10}{50}\right)$ ***A1***

(ii) $\beta = 2\pi - \alpha$ $(= 2\theta)$ ***A1***

$= 2\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)$ ***A1***

***[4 marks]***

(e) (i) $b(x) = x\alpha + 10\beta + 2\sqrt{50^2-(x-10)^2}$ ***A1A1A1***

$= x\left(2\pi - 2\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)\right) + 20\left(\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)\right) + 2\sqrt{50^2-(x-10)^2}$ ***M1A1***

(ii) maximum value of perimeter $= 276$ ***A2***

(iii) perimeter of $200$ cm $b(x) = 200$ ***(M1)***

when $x = 21.2$ ***A1***

***[9 marks]***

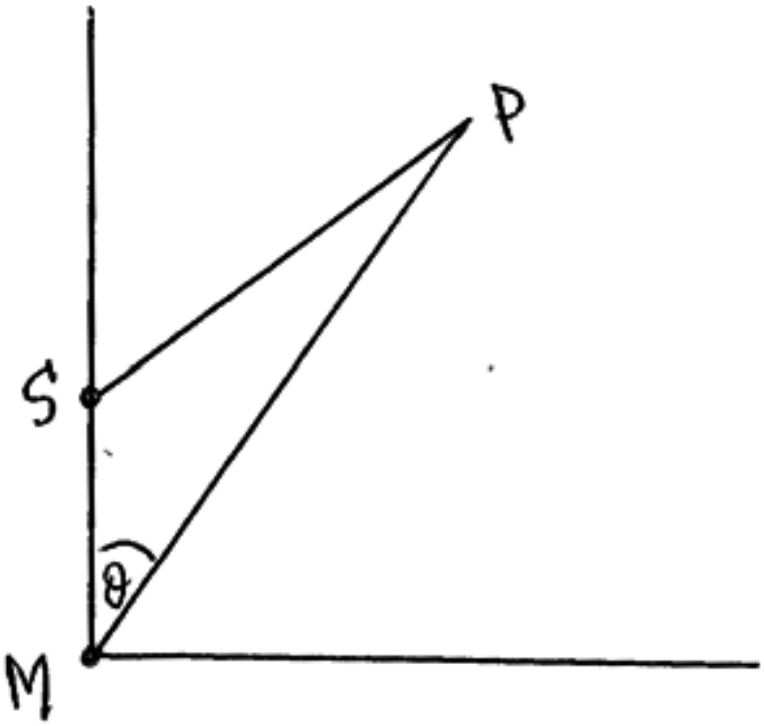
***Total [18 marks]***

**38a.** *[4 marks]*

A ship, S, is 10 km north of a motorboat, M, at 12.00pm. The ship is travelling northeast with a constant velocity of $20\ \text{km hr}^{-1}$. The motorboat wishes to intercept the ship and it moves with a constant velocity of $30\ \text{km hr}^{-1}$ in a direction $\theta$ degrees east of north. In order for the interception to take place, determine

the value of $\theta$.

## Markscheme




let the interception occur at the point P, $t$ hrs after 12:00

then, SP = 20$t$ and MP = 30$t$ ***A1***

using the sine rule,

$\frac{\text{SP}}{\text{MP}} = \frac{2}{3} = \frac{\sin\theta}{\sin 135}$ ***M1A1***

whence $\theta = 28.1$ ***A1***

***[4 marks]***

**38b.** *[5 marks]*

the time at which the interception occurs, correct to the nearest minute.

## Markscheme

using the sine rule again,

$\frac{\text{MP}}{\text{MS}} = \frac{\sin 135}{\sin(45-28.1255\ldots)}$ ***M1A1***

$30t = 10 \times \frac{\sin 135}{\sin 16.8745\ldots}$ ***M1***

$t = 0.81199\ldots$ ***A1***

the interception occurs at 12:49 ***A1***

***[5 marks]***